

GENERALIZED SPIN REPRESENTATIONS AND REDUCTIVE FINITE-DIMENSIONAL QUOTIENTS OF MAXIMAL COMPACT SUBALGEBRAS OF KAC–MOODY ALGEBRAS

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ABSTRACT. We introduce the notion of a generalized spin representation of the maximal compact subalgebra \mathfrak{k} of a simply laced Kac–Moody algebra \mathfrak{g} in order to show that every such non-abelian \mathfrak{k} has a non-trivial finite-dimensional semisimple quotient.

1. INTRODUCTION

During the last decade the family of Kac–Moody algebras $E_n(\mathbb{R})$ has received considerable attention because of its importance in M-theory [DB06], [GN95], [KNP07], [Pal08], [Wes01]. By [DKN06a], [DBHP06] the (so-called) maximal compact subalgebra $\mathfrak{k} = \text{Fix } \omega$ of the real split Kac–Moody algebra $\mathfrak{g} = E_{10}(\mathbb{R})$ with respect to the Cartan–Chevalley involution ω admits a 32-dimensional complex representation which extends the spin representation of its regular subalgebra $\mathfrak{so}_{10}(\mathbb{R})$. This implies that the (infinite-dimensional) Lie algebra \mathfrak{k} has a non-trivial finite-dimensional quotient, in fact a semisimple finite-dimensional quotient (see Corollary 4.10 below). In contrast, maximal compact subalgebras of finite-dimensional simple real Lie algebras are either simple or a direct sum of two isomorphic simple algebras.

In this paper we show that the existence of non-trivial semisimple finite-dimensional representations is not peculiar to the maximal compact subalgebra of $E_{10}(\mathbb{R})$ but is shared by all maximal compact subalgebras of simply laced Kac–Moody algebras over arbitrary fields of characteristic 0. To this end we introduce the notion of a generalized spin representation (Definition 4.3 below), which we inductively show to exist for arbitrary simply laced diagrams (Theorem 4.7 below) and which, in the case of formally real fields, affords a reductive and often even a semisimple image (Corollary 4.12 below).

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2. PRELIMINARIES

In this section we collect several basic facts about Kac–Moody algebras. We refer the reader to [Kac94, chapter 1] and [Kum02, chapter 1] for proofs.

2.1. Kac–Moody algebras. Let k be a field of characteristic 0, let $A = (a_{ij}) \in \mathbf{Z}^{n \times n}$ be a **generalized Cartan matrix** and let $\mathfrak{g} = \mathfrak{g}_A$ denote the corresponding **Kac–Moody algebra** over k . This means that

$$a_{ii} = 2, \quad a_{ij} \leq 0 \quad \text{and} \quad a_{ij} = 0 \Leftrightarrow a_{ji} = 0,$$

while \mathfrak{g} is the quotient of the free Lie algebra over k generated by e_i, f_i, h_i ($i = 1, \dots, n$) subject to the relations

$$\begin{aligned} [h_i, h_j] &= 0, [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j, \\ [e_i, f_j] &= 0, [e_i, f_i] = h_i, (\text{ad } e_i)^{-a_{ij}+1}(e_j) = 0, (\text{ad } f_i)^{-a_{ij}+1}(f_j) = 0 \text{ for } i \neq j. \end{aligned}$$

A generalized Cartan matrix is called **simply laced** if the off-diagonal entries of A are either 0 or -1 . In this case, the **diagram** of \mathfrak{g}_A is the graph $D = (V, E)$ on vertices v_1, \dots, v_n with v_i and v_j connected by an edge if and only if $a_{ij} = -1$.

Let $\mathfrak{h} := \langle h_1, \dots, h_n \rangle$, $\mathfrak{n}_+ := \langle e_1, \dots, e_n \rangle$ and $\mathfrak{n}_- := \langle f_1, \dots, f_n \rangle$ denote the standard subalgebras of \mathfrak{g} . Then there is a decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (for this and the other basic facts about Kac–Moody algebras we will use, we refer the reader to [Kum02, chapter 1] or [Kac94, chapter 1]). Let $Q := \bigoplus_{i=1}^n \mathbb{Z}v_i$ denote a free \mathbb{Z} -module of rank n and $Q_+ := \bigoplus_{i=1}^n \mathbb{N}v_i$. Then \mathfrak{g} has a Q -grading by declaring

$$\deg h_i := 0, \quad \deg e_i := v_i, \quad \deg f_i := -v_i$$

for $i = 1, \dots, n$, i.e.

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha \quad \text{and} \quad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

Let $\Delta := \{\alpha \in Q \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\}$. Then $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ := \Delta \cap Q_+$ and $\Delta_- := -\Delta_+$. An element $\alpha \in \Delta$ is called a **root** and \mathfrak{g}_α a **root space**. A root $\alpha \in \Delta$ is called **positive** if it belongs to Δ_+ , otherwise **negative**. A root of the form $\alpha = \pm v_i$ is called **simple**.

The **extended Weyl group** $W^* \leq \text{Aut } \mathfrak{g}$ is defined as $W^* := \langle s_i^* \mid i = 1, \dots, n \rangle$, where $s_i^* := \exp \text{ad } f_i \cdot \exp \text{ad } -e_i \cdot \exp \text{ad } f_i$ (cf. [Kac94, §3.8]). For $\alpha \in \Delta$ and $w \in W^*$ there exists a unique $w \cdot \alpha \in \Delta$ such that $w \cdot \mathfrak{g}_\alpha = \mathfrak{g}_{w \cdot \alpha}$. A root α is called **real** if there is a $w \in W$ such that $w \cdot \alpha$ is simple, otherwise it is called **imaginary**. Let Δ^{re} denote the set of real roots and Δ^{im} the set of imaginary roots.

For $\alpha = \sum_{i=1}^n a_i v_i \in \Delta$ the **height** of α is defined as $\text{ht } \alpha := |\sum_{i=1}^n a_i|$. For $n \in \mathbb{N}$ let

$$(\mathfrak{n}_+)_n := \bigoplus_{\substack{\alpha \in \Delta^+ \\ \text{ht } \alpha = n}} \mathfrak{g}_\alpha.$$

This is a \mathbb{Z} -grading of \mathfrak{n}_+ .

2.2. The maximal compact subalgebra. Let \mathfrak{g} be a Kac–Moody algebra over a field k of characteristic 0. Let $\omega \in \text{Aut}(\mathfrak{g})$ denote the **Cartan–Chevalley involution** characterized by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h_i) = -h_i$. Let $\mathfrak{k} := \{X \in \mathfrak{g} \mid \omega(X) = X\}$ denote the fixed point subalgebra. Then the subalgebra $\mathfrak{k} = \mathfrak{k}(\mathfrak{g})$ is called the **maximal compact subalgebra** of \mathfrak{g} .

For example, if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, then $\omega(A) = -A^T$ and $\mathfrak{k} = \mathfrak{so}_n(\mathbb{R})$. In this case, $\mathfrak{so}_n(\mathbb{R})$ is the Lie algebra of the maximal compact subgroup $\text{SO}_n(\mathbb{R})$ of $\text{SL}_n(\mathbb{R})$, which explains the terminology. See also [Kna02, Section IV.4].

A theorem of Berman’s allows to give a presentation of \mathfrak{k} .

Theorem 2.1 (cf. [Ber89, Theorem 1.31]). *Let k be a field of characteristic 0. Let $A \in \mathbb{Z}^{n \times n}$ be a simply laced generalized Cartan matrix, let \mathfrak{g}_A denote the corresponding Kac–Moody algebra and let \mathfrak{k} denote the maximal compact subalgebra of \mathfrak{g} .*

Then \mathfrak{k} is isomorphic to the quotient of the free Lie algebra over k generated by X_1, \dots, X_n subject to the relations

$$\begin{aligned} [X_i, [X_i, X_j]] &= -X_j && \text{if the vertices } v_i, v_j \text{ are connected by an edge} \\ [X_i, X_j] &= 0 && \text{otherwise} \end{aligned}$$

via the map $X_i \mapsto e_i - f_i$.

Proof. Let $\eta \in \text{Aut } \mathfrak{g}$ denote the involution characterized by $\eta(e_i) = f_i$, $\eta(f_i) = e_i$ and $\eta(h_i) = -h_i$ and let $\mathfrak{l} := \text{Fix } \eta$ denote the subalgebra of fixed points of η . By [Ber89, Theorem 1.31], \mathfrak{l} is isomorphic to the quotient of the free Lie algebra over k generated by Y_1, \dots, Y_n subject to the relations

$$\begin{aligned} [Y_i, [Y_i, Y_j]] &= Y_j && \text{if the vertices } v_i, v_j \text{ are connected by an edge} \\ [Y_i, Y_j] &= 0 && \text{otherwise} \end{aligned}$$

via the map $Y_i \mapsto e_i + f_i$.

Let $I := \sqrt{-1}$ denote a square root of -1 and let $L := k(I)$, $\mathfrak{g}_L := \mathfrak{g} \otimes_k L$. Note that there is an algebra automorphism $\varphi \in \text{Aut}(\mathfrak{g}_L)$ determined by $e_i \mapsto I \cdot e_i$, $f_i \mapsto -I \cdot f_i$ and $h_i \mapsto h_i$. Then φ conjugates η to ω , i.e. $\omega = \varphi^{-1} \circ \eta \circ \varphi$, and hence the subalgebras $\text{Fix } \eta$ and $\text{Fix } \omega$ are isomorphic over L . As X_i is mapped to $I \cdot Y_i$ under this isomorphism, the conclusion follows. \square

3. SOME ALGEBRAIC PROPERTIES OF \mathfrak{k}

In this section we collect some consequences of Berman's presentation of the maximal compact subalgebra of a Kac–Moody algebra.

3.1. Automorphisms. For $i = 1, \dots, n$ let $\varepsilon_i \in \{\pm 1\}$. Then there is an automorphism φ_ε of \mathfrak{k} characterized by $\varphi(X_i) = \varepsilon_i X_i$. We call such an automorphism a **sign automorphism**.

If π is an automorphism of the diagram D of \mathfrak{g} , i.e. π is a permutation of the vertices of D which preserves adjacency, then there is an induced automorphism φ_π of \mathfrak{k} which satisfies $\varphi_\pi(X_i) = X_{\pi(i)}$. Such an automorphism is called a **graph automorphism**.

Lemma 3.1. *Let \mathfrak{g} be a Kac–Moody algebra over a field k of characteristic 0.*

- (a) *For $i = 1, \dots, n$, the element $s_i^* \in W^*$ commutes with ω .*
- (b) *Every $w \in W^*$ induces an automorphism of \mathfrak{k} .*
- (c) *If the diagram $D = (V, E)$ of \mathfrak{g} is simply laced, the automorphism s_i induced by s_i^* satisfies $s_i(X_i) = X_i$, $s_i(X_j) = X_j$ if $(i, j) \notin E$ and $s_i(X_j) = [X_i, X_j]$ if $(i, j) \in E$.*

Proof. (a) As \mathfrak{g} is generated by e_j, f_j for $j = 1, \dots, n$, it suffices to show that $\omega \circ s_i^* \circ \omega$ agrees with s_i^* on these generators. To this end, let $Y_i := \langle e_i, f_i \rangle \cong \mathfrak{sl}_2(k)$ and consider the Y_i -module

$$M_j := \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}_{\pm \alpha_j + r \alpha_i}.$$

Note that M_j is invariant under ω and that each non-trivial root space $\mathfrak{g}_{\pm \alpha_j + r \alpha_i}$ is one-dimensional. Moreover, $\mathfrak{g}_{\alpha_j + r \alpha_i} = 0$ for $r < 0$ or $r \geq n := -a_{ij}$ and similarly $\mathfrak{g}_{-\alpha_j + r \alpha_i} = 0$ for $r > 0$ or $r \leq -n$.

The action of ω on M_j satisfies $\omega(\mathfrak{g}_{\pm \alpha_j + r \alpha_i}) = \mathfrak{g}_{\mp \alpha_j - r \alpha_i}$. Similarly, $s_i^*(\mathfrak{g}_{\alpha_j + r \alpha_i}) = \mathfrak{g}_{\alpha_j + (n-r) \alpha_i}$ for $0 \leq r \leq n$ and $s_i^*(\mathfrak{g}_{-\alpha_j - r \alpha_i}) = \mathfrak{g}_{-\alpha_j - (n-r) \alpha_i}$ for $0 \leq r \leq n$. From this description the claim follows as M_j contains e_j and f_j .

- (b) By (a), each s_i^* stabilizes \mathfrak{k} . The claim therefore follows immediately from [Kac94, Lemma 3.8(b)].
- (c) A calculation shows that $s_i^*(e_i) = -f_i$ and $s_i^*(e_j) = [e_i, e_j]$ if $(i, j) \in E$ and $s_i^*(e_j) = e_j$ if $(i, j) \notin E$. (Cf. also [Kac94, Lemma 3.8(a)].)

□

For $w \in W^*$, the induced automorphism $\pi(w) \in \text{Aut } \mathfrak{k}$ is called a **Weyl group automorphism**.

Remark 3.2. (a) Let $\varphi: \mathfrak{n}_+ \rightarrow \mathfrak{k}, x \mapsto x + \omega(x)$ denote the canonical k -linear bijection (cf. [Ber89, p. 3169]), and write $\mathfrak{k}_\alpha := \varphi(\mathfrak{g}_\alpha)$. It follows that $\pi(s)(\mathfrak{k}_\alpha) = \mathfrak{k}_{s \cdot \alpha}$. It is then clear by induction that for a positive real root α there is a Weyl group automorphism w and a positive simple root α_i such that w maps \mathfrak{k}_α to some $\mathfrak{k}_{\alpha_i} = k \cdot X_i$.

(b) The set of subspaces $\{\mathfrak{k}_\gamma \mid \gamma \in \Delta^{re} \cap \Delta_+\}$ is left invariant by the group of Weyl group automorphisms. It can be identified with the walls of the Coxeter complex of the Weyl group W .

Remark 3.3. For i, j in the same connected component of the diagram of \mathfrak{k} there is an automorphism such that $\varphi(X_i) = X_j$. This is because, if (i, j) is an edge, then $s_i s_j(X_i) = s_i([X_j, X_i]) = [[X_i, X_j], X_i] = X_j$; thus, the claim follows by induction.

This can be used as follows: Let \mathfrak{g} be the Kac–Moody algebra of type AE_2 (see the appendix for our notation concerning Dynkin diagrams). Then the generator X_4 is contained in a subalgebra isomorphic to A_2^+ . Indeed, let φ be a Weyl group automorphism such that $\varphi(X_3) = X_4$. Then $\varphi(\langle X_1, X_2, X_3 \rangle)$ is as required.

3.2. The center of \mathfrak{k} . Except in the case that \mathfrak{g} has a factor of type A_1 , the maximal compact subalgebra is perfect and centerfree.

Proposition 3.4. *Let k be a field of characteristic 0, let D be a simply laced diagram without isolated nodes, let \mathfrak{g} be a Kac–Moody algebra with diagram D and let \mathfrak{k} denote its maximal compact subalgebra. The the following assertions hold:*

- (a) \mathfrak{k} is perfect, i.e. $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$.
- (b) The center of \mathfrak{k} is trivial: $C(\mathfrak{k}) = \{0\}$.

Proof. (a) By assumption, for each generator X_i of \mathfrak{k} there is some j such that the vertices i, j are connected by an edge. From the relation $[X_j, [X_j, X_i]] = -X_i$ (cf. Theorem 2.1) it follows that every X_i is contained in the derived subalgebra.

- (b) Suppose that there is $c \in C(\mathfrak{k}) \setminus \{0\}$. Write $c = x + \omega(x)$, where $x = x_r + x_{r+1} + \dots + x_s \in \mathfrak{n}_+$, $x_i \in (\mathfrak{n}_+)_i$, $r \leq s$ and $x_r, x_s \neq 0$. Suppose first that $r > 1$. As $c \in C(\mathfrak{k})$, one has $[X_i, c] = 0$ for all i . This implies that $[f_i, x_r] = 0$ for all i since $[f_i, x_r]$ is the homogeneous component of $[X_i, c]$ of degree $r - 1$. By [Kac94, Lemma 1.5] this yields the contradiction $x_r = 0$.

If $r = 1$, write $x_1 = \sum_i a_i e_i$ with a_i in k and let i_0 be such that $a_{i_0} \neq 0$. Let ψ be the sign automorphism given by $X_{i_0} \mapsto X_{i_0}$ and $X_j \mapsto -X_j$ for $j \neq i_0$. Then $c + \psi(c) \in C(\mathfrak{k})$ and its homogeneous component of degree 1 equals $2a_{i_0} e_{i_0}$. Replacing c by $\frac{1}{2a_{i_0}}(c + \psi(c))$ we may assume that $x_1 = e_{i_0}$. By adding suitable images of sign automorphisms to c and rescaling, we may additionally assume that $x_3 = 0$. This is because an element of the form $[e_a, [e_b, e_c]]$ is 0 if a, b, c are not pairwise distinct, while for distinct a, b, c it can be erased by using a sign automorphism which preserves X_{i_0} and, for example, sends X_a to $-X_a$. Since the vertex i_0 is not isolated, there is some j such that $[X_{i_0}, X_j] \neq 0$. But then $[c, X_j] \neq 0$ since $[e_{i_0}, e_j]$ is its homogeneous component of degree 2.

□

3.3. A contraction of \mathfrak{k} . Let \mathfrak{g} be a simply laced Kac–Moody algebra over \mathbb{R} with Chevalley generators e_i, f_i, h_i ($i = 1, \dots, n$). For $\varepsilon > 0$ define ω_ε to be the Lie algebra automorphism satisfying

$$\omega_\varepsilon(e_i) = -\varepsilon f_i, \quad \omega_\varepsilon(f_i) = -\frac{1}{\varepsilon} e_i, \quad \omega_\varepsilon(h_i) = -h_i.$$

and set $\mathfrak{k}_\varepsilon := \text{Fix } \omega_\varepsilon$. Note $\mathfrak{k} = \mathfrak{k}_1$ and that $X_i^\varepsilon := e_i - \varepsilon f_i \in \mathfrak{k}_\varepsilon$ for $i = 1, \dots, n$. A calculation shows that if the nodes i, j are connected by an edge, then $[X_i^\varepsilon, [X_i^\varepsilon, X_j^\varepsilon]] = -\varepsilon X_j^\varepsilon$. A straightforward adaption of Berman’s proof of [Ber89, Theorem 1.31] shows that \mathfrak{k}_ε is generated by the X_i^ε ’s and that these relations already define the Lie algebra.

Proposition 3.5. *The subalgebra \mathfrak{k}_ε is isomorphic to the quotient of the free Lie algebra over k generated by X_1, \dots, X_n subject to the relations*

$$\begin{aligned} [X_i, [X_i, X_j]] &= -\varepsilon X_j & \text{if the vertices } v_i, v_j \text{ are connected by an edge} \\ [X_i, X_j] &= 0 & \text{otherwise} \end{aligned}$$

via the map $X_i \mapsto e_i - \varepsilon f_i$.

Note that if we set $\varepsilon = 0$ in the above presentation, the resulting algebra is isomorphic to \mathfrak{n}_+ . This means that \mathfrak{n}_+ is a **contraction** of the maximal compact subalgebra $\mathfrak{k} = \mathfrak{k}_1$ in the sense of [FdM06].

3.4. Quotients. Let k be a field of characteristic 0 and \mathfrak{g} a Kac–Moody algebra over k with simply laced diagram D . Due to the Coxeter-like presentation of the maximal compact subalgebra \mathfrak{k} it is possible to exhibit quotients of \mathfrak{k} if D has a certain shape.

For a graph D , let $\mathfrak{k}(D)$ denote the maximal compact subalgebra of the Kac–Moody algebra \mathfrak{g} over k with diagram D .

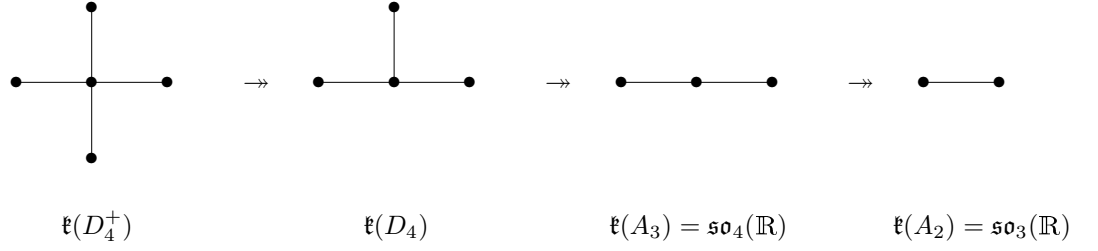
Construction 3.6. Suppose that there are distinct vertices v_i, v_j such that any vertex v_r distinct from v_i, v_j is connected to v_i if and only if v_r is connected to v_j .

- (a) If v_i and v_j are not connected by an edge, let D' be the diagram obtained from D by deleting the vertex v_j . Let $\mathfrak{k}' := \mathfrak{k}(D')$ and X'_1, \dots, X'_n its Berman generators. Then there is a well-defined epimorphism of Lie algebras $\varphi: \mathfrak{k} \rightarrow \mathfrak{k}'$ determined by $\varphi(X_r) := X'_r$ for $r \neq j$ and $\varphi(X_j) := X'_i$.
- (b) If v_i and v_j are connected by an edge, let D' be the diagram obtained from D by deleting all edges emanating from v_j except for the edge (v_i, v_j) . As above, let $\mathfrak{k}' := \mathfrak{k}(D')$ and X'_1, \dots, X'_n its Berman generators. Then there is a well-defined epimorphism of Lie algebras $\varphi: \mathfrak{k} \rightarrow \mathfrak{k}'$ determined by $\varphi(X_r) := X'_r$ for $r \neq j$ and $\varphi(X_j) := [X'_i, X'_j]$.

Example 3.7. (a) The preceding discussion gives a sequence of epimorphisms of real Lie algebras

$$\mathfrak{k}(D_4^+) \twoheadrightarrow \mathfrak{k}(D_4) \twoheadrightarrow \mathfrak{k}(A_3) = \mathfrak{so}_4(\mathbb{R}) \twoheadrightarrow \mathfrak{k}(A_2) = \mathfrak{so}_3(\mathbb{R}).$$

This sequence can be extended further: Let Γ_n denote the star graph given by $\Gamma = (\{1, \dots, n\}, \{\{1, k\} \mid 2 \leq k \leq n\})$ and let \mathfrak{k}_n denote the maximal compact subalgebra of the Kac–Moody algebra \mathfrak{g}_n with Dynkin diagram Γ_n . Then there are epimorphisms $\mathfrak{k}_n \rightarrow \mathfrak{k}_{n-1}$.



- (b) Denoting by K_4 the complete graph on four vertices, there similarly is a sequence of epimorphisms $\mathfrak{k}(K_4) \twoheadrightarrow \mathfrak{k}(AE_2) \twoheadrightarrow \mathfrak{k}(A_4)$.

4. GENERALIZED SPIN REPRESENTATIONS

We recall first the extension of the spin representation of $\mathfrak{k}(\mathfrak{sl}_{10}(\mathbb{R}))$ to $\mathfrak{k}(E_{10}(\mathbb{R}))$ as described in [DKN06a], [DBHP06], [Kau04].

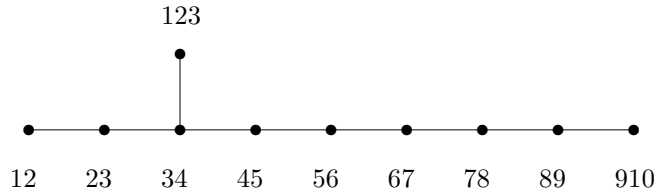
Example 4.1. Let V be a k -vector space and $q: V \rightarrow k$ a quadratic form with associated bilinear form b . Then the **Clifford algebra** $C := C(V, q)$ is defined as $C := T(V)/\langle vw + wv - b(v, w) \rangle$ where $T(V)$ is the tensor algebra of V .

Now let $V = \mathbb{R}^{10}$ with standard basis vectors v_i , let $q = -(x_1^2 + \dots + x_{10}^2)$ and let $C = C(V, q)$. Note that in C we have

$$v_i^2 = -1 \text{ and } v_i v_j = -v_j v_i.$$

Since C is an associative algebra, it becomes a Lie algebra by setting $[A, B] := AB - BA$.

Let the diagram of $E_{10}(\mathbb{R})$ be labelled as follows.



Associate to the Berman generator X_i corresponding to the node v_i the element $\frac{1}{2}v_I$ of C , where I is the label of v_i . For example $X_1 \mapsto \frac{1}{2}v_1v_2$ and $X_2 \mapsto \frac{1}{2}v_1v_2v_3$ with the labelling of the vertices as in the appendix.

One checks easily that this map extends to a Lie algebra homomorphism $\rho: \mathfrak{k} \rightarrow C$, i.e. that the defining relations are respected. The restriction of ρ to the maximal compact subalgebra of the A_9 -subdiagram, $\mathfrak{k}(A_9) = \mathfrak{so}_{10}(\mathbb{R})$, can be shown to coincide with the spin representation of \mathfrak{so}_{10} (see e.g. [FH91, Chapter 20]), i.e. ρ extends this representation. Now C splits over \mathbb{C} , i.e. $C \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^{32 \times 32}$ ([FH91, Lemma 20.9]). Hence ρ affords a 32-dimensional complex representation of $\mathfrak{k}(E_{10}(\mathbb{R}))$. A calculation shows that $\text{im } \rho$ is the linear span of all elements of the form v_I , where $I \subseteq \{1, \dots, 10\}$ with $|I| \in \{2, 3, 6, 7, 10\}$. Therefore $\dim \text{im } \rho = 496$ and it can be shown that $\text{im } \rho \cong \mathfrak{so}_{32}(\mathbb{R})$, see [DKN06b].

Observation. Let $\iota \in \text{Aut } C$ denote the involution induced by $v \mapsto -v$. Let $C_0 := \text{Fix } \iota$ and $C_1 := \{w \in C \mid \iota(w) = -w\}$ denote the even and the odd part of C . Then C_0 and C_1 are invariant under the spin representation of \mathfrak{so}_{10} since $\text{im } \rho \subseteq C_0$ and these are irreducible non-isomorphic representations of \mathfrak{so}_{10} ([FH91, Chapter 20]). The remaining Berman generator of $\mathfrak{k}(E_{10})$ is sent to an element

which interchanges C_0 and C_1 . Moreover, each $A_i := \rho(X_i)$ satisfies $A_i^2 = -\frac{1}{4} \text{id}$.

This observation is the basis for our construction of generalized spin representations.

Remark 4.2. Let $\rho: \mathfrak{so}_{10}(\mathbb{R}) \rightarrow \mathbb{C}^{n \times n}$ be a representation. To extend ρ to a representation of $\mathfrak{k}(E_{10})$, it suffices to find a matrix $X \in \mathbb{C}^{n \times n}$ such that for $A_i := \rho(X_i)$, the following equations are satisfied (where we use the labelling of the diagram of E_{10} as given in the appendix):

$$\begin{aligned} [A_i, X] &= 0 & \text{for } 1 \leq i \leq 10, i \neq 4, \\ [A_4, [A_4, X]] &= -X, \\ [X, [X, A_4]] &= -A_4. \end{aligned}$$

The first two sets of equations define a linear subspace, the third set of equations yields a family of quadratic equations. With the help of a Gröbner basis one can compute that in case of the generalized spin representation, this variety is isomorphic to \mathbb{C}^\times , i.e., the extension is unique up to a scalar.

Throughout this section, let k be a field of characteristic 0, let \mathfrak{g} a Kac–Moody algebra over k with simply laced diagram and \mathfrak{k} its maximal compact subalgebra. Let $L := k(I)$, where I is a square root of -1 and denote by $\text{id}_s \in L^{s \times s}$ the identity matrix.

Definition 4.3. A representation $\rho: \mathfrak{k} \rightarrow \text{End}(L^s)$ is called a **generalized spin representation** if the images of the Berman generators from Theorem 2.1 satisfy

$$\rho(X_i)^2 = -\frac{1}{4} \text{id}_s \text{ for } i = 1, \dots, n.$$

Remark 4.4. (a) Since ρ is assumed to be a representation, it follows from the defining relations that $\rho(X_i)$ and $\rho(X_j)$ commute if $(i, j) \notin E$, while if $(i, j) \in E$, then $A := \rho(X_i)$ and $B := \rho(X_j)$ anticommute. Indeed we have

$$-B = [A, [A, B]] = A^2 B - 2ABA + BA^2 = -\frac{1}{2}B - 2ABA$$

from which the claim follows after multiplying with $A^{-1} = -4A$.

(b) Conversely, suppose that there are matrices $A_i \in L^{s \times s}$ satisfying

- (i) $A_i^2 = -\frac{1}{4} \cdot \text{id}_s$,
- (ii) $A_i A_j = A_j A_i$ if $(i, j) \notin E$,
- (iii) $A_i A_j = -A_j A_i$ if $(i, j) \in E$.

Then the assignment $X_i \mapsto A_i$ gives rise to a representation of \mathfrak{k} .

Remark 4.5. Let ρ be a generalized spin representation of \mathfrak{k} and set $A_i := 2I \cdot \rho(X_i)$. Let W be a Coxeter group defined by the presentation

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} = 2$ if $(i, j) \notin E$, while $m_{ij} = 4$ if $(i, j) \in E$. Then the assignment $s_i \mapsto A_i$ gives a representation of W .

Let \mathcal{C} denote the class of all generalized spin representation of \mathfrak{k} . We check some closure properties of \mathcal{C} .

Proposition 4.6. (a) \mathcal{C} is closed under direct products, quotients, duals and taking subrepresentations.

- (b) If $\rho_1, \rho_2, \rho_3 \in \mathcal{C}$, then so is $\rho_4: X_i \mapsto 4\rho_1(X_i) \otimes \rho_2(X_i) \otimes \rho_3(X_i)$.
- (c) If $\rho \in \mathcal{C}$ and φ is either a sign, graph or Weyl group automorphism of \mathfrak{k} , then $\rho \circ \varphi \in \mathcal{C}$.

Proof. The first and the second point can be easily verified. The third point is clear when π is a graph or a sign automorphism. For the remaining assertion, it suffices to prove that for $i, j \in \{1, \dots, n\}$ one has that $\rho(s_i^*(X_j))^2 = -\frac{1}{4} \text{id}_s$. Now $s_i^*(X_j)$ is either equal to X_j , for which the claim is trivially true, or is equal to $[X_i, X_j]$. In this case write $A := \rho(X_i)$ and $B := \rho(X_j)$. Then A and B anticommute and $A^2 = B^2 = -\frac{1}{4} \text{id}_s$, from which it follows that

$$(AB - BA)^2 = 4(AB)^2 = -4A^2B^2 = -\frac{1}{4} \text{id}_s.$$

□

Write $\mathfrak{k}_{\leq r} := \langle X_1, \dots, X_r \rangle$.

Theorem 4.7. *Let $1 \leq r < n$. Let $\rho: \mathfrak{k}_{\leq r} \rightarrow \text{End}(L^s)$ be a generalized spin representation.*

- (a) *If X_{r+1} centralizes $\mathfrak{k}_{\leq r}$, then ρ can be extended to a generalized spin representation $\rho': \mathfrak{k}_{\leq r+1} \rightarrow \text{End}(L^s)$ by setting $\rho'(X_{r+1}) := \frac{1}{2}I \text{id}_s$.*
- (b) *If X_{r+1} does not centralize $\mathfrak{k}_{\leq r}$, then ρ can be extended to a generalized spin representation $\rho': \mathfrak{k}_{\leq r+1} \rightarrow \text{End}(L^s \oplus L^s)$. More precisely,*

$$\rho'|_{\mathfrak{k}_{\leq r}} = \rho \oplus \rho \circ s_0,$$

where s_0 is a suitable sign automorphism, and

$$\rho'(X_{r+1}) := \frac{1}{2}I \text{id}_s \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. If X_{r+1} centralizes $\mathfrak{k}_{\leq r}$, it is clear that ρ' is well-defined and that $\rho'(X_r)^2 = -\frac{1}{4} \text{id}_s$.

In the second case, let s_0 be the sign automorphism defined as follows:

$$s_0(X_i) := \begin{cases} X_i & \text{if } (i, r+1) \notin E \\ -X_i & \text{if } (i, r+1) \in E \end{cases}$$

Define ρ' as in the statement of the theorem. Then it is clear that $\rho'|_{\mathfrak{k}_{\leq r}}$ is a generalized spin representation which extends ρ . It is easy to check that $\rho'(X_i)$ commutes with $\rho'(X_{r+1})$ if $(i, r+1) \notin E$ and that $\rho'(X_i)$ anticommutes with $\rho'(X_{r+1})$ if $(i, r+1) \in E$. Again it is clear that ρ' is a generalized spin representation. □

Remark 4.8. A similar inductive construction of the basic spin representations of the symmetric group has independently been obtained by Maas [Maa10]. It is likely that by a combination of the methods of [Maa10] and of the present article a similar construction of generalized (basic) spin representations is possible for any simply laced Coxeter group.

For a graph $G = (V, E)$ a subset $M \subseteq V$ is called **independent** if the subgraph of G induced on M does not contain any edges, i.e. if no two elements m_1, m_2 in M are connected by an edge.

Corollary 4.9. *Let r denote the size of a maximal independent set of the diagram of \mathfrak{g} . Then there exists a 2^{n-r} -dimensional irreducible generalized spin representation of \mathfrak{k} .*

Proof. Up to a change of labelling the set $M := \{v_1, \dots, v_r\}$ is maximal independent. The map $\rho: \mathfrak{k}_{\leq r} \rightarrow \text{End}(L^1): X_i \mapsto \frac{1}{2}I$ is a generalized spin representation. By Theorem 4.7 the representation ρ can be extended inductively to a generalized spin representation of \mathfrak{k} ; the dimension doubles at each step because M was assumed to be maximal independent. □

Corollary 4.10. \mathfrak{k} admits a non-trivial finite-dimensional reductive quotient.

Proof. Note that \mathfrak{k} is abelian if and only if the diagram D of \mathfrak{g} does not contain an edge; cf. Theorem 2.1 and Proposition 3.4(b) (applied to a non-trivial connected component of D). In this case \mathfrak{g} and \mathfrak{k} are finite-dimensional.

It suffices to prove that, if \mathfrak{k} is non-abelian, then it admits a non-trivial finite-dimensional semisimple quotient. Let ρ denote a generalized spin representation as constructed in Corollary 4.9. By the Levi decomposition of a Lie algebra in characteristic 0 it suffices to show that $\text{im}(\rho)$ is not solvable. However, this is clear, as for an edge $(i, j) \in E$ the subalgebra $\rho(\langle X_i, X_j \rangle) \subseteq \text{im}(\rho)$ is not solvable. \square

Remark 4.11. It is possible to construct finite-dimensional quotients of \mathfrak{k} with non-reductive image, even in the case when the diagram is of indefinite type. For example, let \mathfrak{k} be the maximal compact subalgebra of the Kac–Moody algebra \mathfrak{g} with diagram the complete graph on $\{1, 2, 3, 4\}$ with the edge $(1, 4)$ removed. Let $I \leq \mathfrak{k}$ denote the ideal generated by $X_{123123123}, X_{234}$, where $X_{i_1, i_2, \dots, i_n} := [X_{i_1}, X_{i_2, \dots, i_n}]$. Then the following Magma script shows that \mathfrak{k}/I is 42-dimensional with a 15-dimensional solvable radical.

```
k:=4;
L<x1,x2,x3,x4>:= FreeLieAlgebra(RationalField(), k);
pp:= { [1,2],[1,3],[2,3],[2,4],[3,4] };
R:= [ ];
g:= [ L.i : i in [1..k] ];

for i in [1..k] do
  for j in [i+1..k] do
    if [i,j] in pp then
      a:= g[i]*(g[i]*g[j])+g[j];
      Append( ~R, a );
      Append( ~R, g[j]*(g[j]*g[i])+g[i] );
    else
      Append( ~R, g[i]*g[j] );
    end if;
  end for;
end for;

R:=R cat [(x1,(x2,(x3,(x1,(x2,(x3,(x1,(x2,x3))))))),(x2,(x3,x4))];

K, G, B, f := LieAlgebra(R);
K;
SolubleRadical(K);

// Lie Algebra of dimension 42 with base ring Rational Field
// Lie Algebra of dimension 15 with base ring Rational Field
```

In case k is formally real, one can show more using a standard complexification argument:

Corollary 4.12. *Let k be formally real, let I be a square root of -1 , let $L := k(I)$ be the corresponding quadratic extension of k , and let ρ be a generalized spin representation of \mathfrak{k} on an m -dimensional L -vector space V as constructed in Corollary 4.9. Then $\text{im}(\rho)$ is reductive. In particular, if the underlying diagram does not contain any isolated nodes, then $\text{im}(\rho)$ is semisimple.*

Proof. We follow the proof of [HN91, Theorem III.10.10]. Observe first that there exist a k -basis of V such that $\text{im}(\rho)$ is invariant under $X \mapsto -X^T$. Define

$$\mathfrak{k}_- := \{X \in \text{im}(\rho) \mid X^T = -X\} \quad \text{and} \quad \mathfrak{k}_+ := \{X \in \text{im}(\rho) \mid X^T = X\}.$$

As $X \mapsto -X^T$ is an automorphism, the set \mathfrak{k}_- is a sub-Lie algebra of $\text{im}(\rho)$. We have $[\mathfrak{k}_-, \mathfrak{k}_+] \subseteq \mathfrak{k}_+$ and $[\mathfrak{k}_+, \mathfrak{k}_+] \subseteq \mathfrak{k}_-$.

It suffices to show that the radical \mathfrak{r} of $\text{im}(\rho)$ satisfies $\mathfrak{r} \subseteq Z(\text{im}(\rho))$. As \mathfrak{r} is characteristic, one has $\mathfrak{r} = \mathfrak{r} \cap \mathfrak{k}_- + \mathfrak{r} \cap \mathfrak{k}_+$. We conclude that

$$\tilde{\mathfrak{r}} := \mathfrak{r} \cap \mathfrak{k}_- + I(\mathfrak{r} \cap \mathfrak{k}_+) \subseteq \widetilde{\text{im}(\rho)} := \mathfrak{k}_- + I\mathfrak{k}_+ \subseteq \mathfrak{u}(m, L)$$

is a sub-Lie algebra of $\mathfrak{u}(m, L)$ and an ideal of $\widetilde{\text{im}(\rho)}$. Then $\tilde{\mathfrak{r}}$ is an ideal of the compact (and hence reductive) Lie algebra $\widetilde{\text{im}(\rho)}$. We conclude $\tilde{\mathfrak{r}} \subseteq Z(\widetilde{\text{im}(\rho)})$ and so $\mathfrak{r} \subseteq Z(\text{im}(\rho))$, and $\text{im}(\rho)$ is reductive. The final statement now follows from Proposition 3.4. \square

Example 4.13. Let K_n denote the complete graph on n vertices and let \mathfrak{g}_n denote the Kac–Moody algebra over \mathbb{R} with diagram K_n . Then the maximal compact subalgebra k_n of \mathfrak{g}_n surjects onto $\mathfrak{so}_{n+1}(\mathbb{R})$. Indeed, let C_n denote the Clifford algebra associated to the quadratic form $q = -(x_1^2 + \dots + x_n^2)$ on \mathbb{R}^n with basis vectors (v_i) . Then the assignment $X_i \mapsto \frac{1}{2}v_i$ gives a representation ρ of \mathfrak{k}_n , and a calculation shows that

$$M := \text{im } \rho = \bigoplus_{i=1}^n \mathbb{R}v_i \bigoplus \bigoplus_{i < j} \mathbb{R}v_i v_j.$$

Let Y_1, \dots, Y_n denote the Berman generators of $\mathfrak{so}_{n+1}(\mathbb{R}) = \mathfrak{k}(A_n(\mathbb{R}))$. Then the map $r: Y_i \mapsto \frac{1}{2}v_i v_{i+1}$ for $i = 1, \dots, n-1$ and $r(Y_n) := \frac{1}{2}v_n$ extends to a representation of $\mathfrak{so}_{n+1}(\mathbb{R})$ with image equal to M , from which the claim follows.

As soon as \mathfrak{g} is infinite-dimensional, a generalized spin representation of \mathfrak{k} will have an infinite-dimensional kernel. If \mathfrak{g} is affine, the following proposition describes a large submodule of this kernel. It is an amusing exercise to show that if \mathfrak{g} is of indefinite type, it contains at least one affine subdiagram, so that in this case the proposition applies to each affine subdiagram.

Proposition 4.14. *Let \mathfrak{g} be of affine type and let ρ be a generalized spin representation of the maximal compact subalgebra of \mathfrak{g} . Then $\ker \rho$ contains the ideal generated by \mathfrak{k}_δ .*

Proof. If the diagram is of type A_n^+ , the vertex labelled $n+1$ is joined to the vertices 1 and n . Then \mathfrak{k}_δ is spanned by the element $G := [X_1, [X_2, \dots, [X_n, X_{n+1}] \dots]]$ and those obtained from G by a cyclic shift of the indices. Expanding $\rho(G)$ and using the fact that $\rho(X_i)$ and $\rho(X_j)$ either commute or anti-commute, it follows that $\rho(G) = 0$.

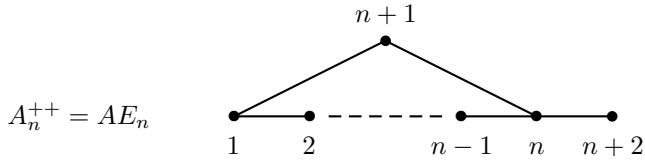
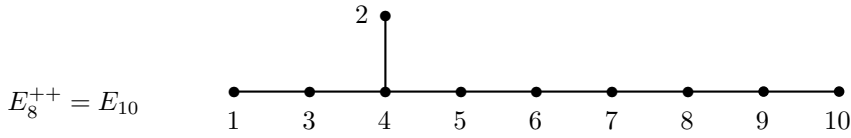
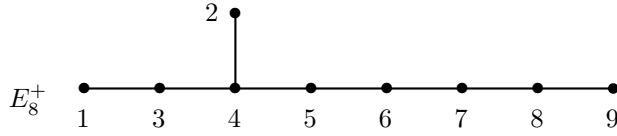
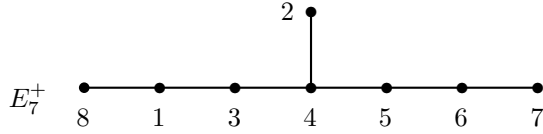
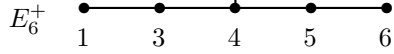
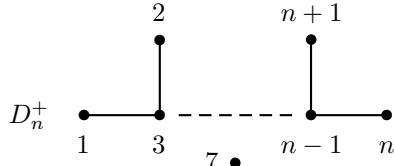
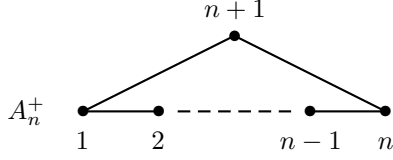
For the remaining diagrams of type D_n^+ and E_6^+, E_7^+ and E_8^+ an inspection of table Aff₁ in [Kac94, p. 54] shows that the extra vertex is joined to a vertex such that the corresponding simple root appears with multiplicity 2 in the decomposition of δ into simple roots. As above, by expanding and using the (anti)-commutativity this implies that any element in the space \mathfrak{k}_δ is killed by ρ . \square

Remark 4.15. A computer calculation shows that the kernel of the generalized spin representation of $\mathfrak{k}(E_{10})(\mathbb{R})$ is already generated by \mathfrak{k}_δ , where δ denotes the fundamental positive imaginary root of the regular subalgebra E_9 . In general, though, the kernel can be larger. For example, the generalized spin representation ρ of $\mathfrak{k}(A_2^+)$ with $\text{im } \rho \cong \mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{so}_3(\mathbb{C}) \times \mathfrak{so}_3(\mathbb{C})$ can be post-composed with a

projection to one of the factors to obtain another generalized spin representation with larger kernel.

5. APPENDIX

We give the list of relevant Dynkin diagrams we use in the main text.



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